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Resolution of stringy singularities by non-commutative algebras

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Abstract: In this paper we propose a unified approach to (topological) string theory on certain singular spaces in their large volume limit. The approach exploits the non-commutative structure of D-branes, so the space is described by an algebraic geometry of non-commutative rings. The paper is devoted to the study of examples of these algebras. In our study there is an auxiliary commutative algebraic geometry of the center of the (local) algebras which plays an important role as the target space geometry where closed strings propagate. The singularities that are resolved will be the singularities of this auxiliary geometry. The singularities are resolved by the non-commutative algebra if the local non-commutative rings are regular. This definition guarantees that D-branes have a well defined K-theory class. Homological functors also play an important role. They describe the intersection theory of D-branes and lead to a formal definition of local quivers at singularities, which can be computed explicitly for many types of singularities. These results can be interpreted in terms of the derived category of coherent sheaves over the non-commutative rings, giving a non-commutative version of recent work by M. Douglas. We also describe global features like the Betti numbers of compact singular Calabi-Yau threefolds via global holomorphic sections of cyclic homology classes.

Keywords: D-branes, Differential and Algebraic Geometry, Non-Commutative Geometry.

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Contents

1. Introduction

D-branes [1, 2] have played a pivotal role in changing our understanding of string theory. Even though D-branes are nonperturbative effects in string theory, they are remarkably simple mainly because they admit a description via boundary conformal field theory. A D-brane configuration defines in some sense an open string field theory, and in open string theories one can derive the closed string states from the knowledge of the open string amplitudes by analyzing the pole structure in the one loop amplitudes.

It has been known for a while that open string theories are inherently non-commutative objects [3], that is, the operation of gluing two open strings distinguishes the ends of the strings at which they are joined. It was not until the advent of Matrix theory [4] that this important feature of open strings was appreciated, as there it was seen that the target space coordinates of D-branes are described by matrices rather than just numbers. More recently in [5] and [6] it has been argued that field theories on non-commutative spaces are low energy limits of open string theories and the origin of non-commutativity has been related to the NS B-field. As well, the classification of topological D-brane charges is given by the topological K-theory of the target space [7, 8, 9], which is in fact a well defined group for any non-commutative algebra[1] (see [10] for example).

More recently we have found that non-commutative algebras give a description of moduli spaces of D-branes on certain orbifold singularities [11], and that this description is sufficient to calculate the Betti numbers of certain singular orbifold Calabi-Yau spaces [12] without computing the topological closed string theory spectrum.

There are compactification spaces of string theory which are singular in the large volume limit as commutative geometries (orbifolds with fixed points for example), whereas the string theory on these spaces is perfectly smooth [13, 14] since one can compute all the correlation functions in the CFT. If one resolves all of the singularities geometrically (either by blow-ups or deformations, see [15, 16, 17] for example) then one can calculate the stringy data for these deformed spaces and take the limit towards the singular geometry to describe the singular space. There are exceptions to this argument: not all singularities can be resolved in this manner, as was shown for a particular example of orbifolds with discrete torsion [18]. There the generic deformation of the Calabi Yau space has conifold singularities which cannot be blown up or deformed away. However, in [12] we showed that these spaces admit an elegant non-commutative description. If one considers matrix theory as a prototype, it is natural to define the target space geometry as that which is seen by D-branes, and ask questions about closed strings later. That is, instead of defining a target space as a closed string theory background, one would prefer to ask what are the allowed configurations of D-branes and infer the closed string background from this data. Naturally, D-branes via their boundary states can carry all of the possible closed string state information, as they are sources for all possible closed string states. In some sense, the full D-brane data can be as complicated as the closed string theory of the background itself, so we opt for a less detailed description in terms of topological low energy degrees of freedom alone.

Our intention in this paper is to make a general proposal for 'resolving' singularities within non-commutative geometry and to understand the D-branes on these

[1]In string theory one can argue that the algebras in questions are $\mathbb{C}^*$ algebras [3], so it is appropriate to consider the K-theory of a $\mathbb{C}^*$ algebra.

spaces. The main idea is that even when one has blown down a cycle to obtain a singularity, one can still have a B field through the shrunken cycle. Geometrically we have

$$\frac{\int_\Sigma Re(\omega)}{\int_\Sigma Im(\omega)} \to \infty \tag{1.1}$$

(where $\omega = B + iJ$ is the complexified Kähler form) so the B field is locally very large in geometric units. From the arguments of ref. [6] this should signal that the geometry of the singularity is well described by a non-commutative geometry.

The basic outline of our program is as follows: one is given a non-commutative complex manifold, that is, a collection of holomorphic non-commutative algebras describing analytic coordinate patches on the complex manifold, and maps on how to glue these algebras. In our case the algebras on each patch $\mathcal{A}_i$ have a center $\mathcal{Z}\mathcal{A}_i$, and the center is glued to give a commutative algebra. The algebraic geometry of this commutative algebra will be identified with the target space where closed strings propagate, which might be singular, and the algebraic geometry of the non-commutative algebras $\mathcal{A}_i$ will give the resolution of those singularities. This is the spirit of [6] where there are two geometries at play, one for closed strings and another one for open strings. In this sense a commutative singularity can be made smooth in a non-commutative sense. We are not making the claim however that *any* singularity can be resolved in this way; those that admit conformal field theory descriptions should admit this type of resolution. The notion of smoothness will be given by an algebraic definition (namely that locally the algebras be regular), and it is intimately tied up to the derived category of coherent sheaves of the algebra (that is, the D-branes are giving us a definition of smoothness). Here we find the approach of [19, 20] very illuminating. One should be able to define the target space of string theory by using the derived category of coherent sheaves (holomorphic B model D-branes), which are the topological branes. This approach has natural advantages. Once we satisfy our definition of smoothness every coherent sheaf will admit a (locally) projective resolution. That is, any brane can be obtained from branes and antibranes on the non-commutative manifold which look like vector bundles, with a choice of tachyon profile. This is desirable because it tells us that the classification of D-branes is given by the algebraic K-theory of the non-commutative space [21]. For orbifolds of a smooth space, the construction reproduces the quiver diagrams for brane fractionation (à la Douglas-Moore [22]), and moreover the K-theory of this algebra is automatically the (twisted) equivariant K-theory of the original commutative algebra with respect to the orbifold group action (see [23, 10] for example). The projective resolution also allows us to compute the number of strings between two branes, and from this data one can define the intersection number of pairs of branes.

In this paper we will only ask questions that do not depend on the target space metric and instead work with just configurations of BPS branes, which are equally

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well described by the topologically twisted string theories associated to a target space. This is also the natural setting for studying mirror symmetry and quantum cohomology [24], which make this coarser description a good place to find everything which is calculable at a generic point in moduli space.

We have chosen to present our proposal through examples, and we will deal with these as if they are generic rather than proceed with an abstract theory. Our purpose is to find a constructive approach to these algebras (the ones that resolve singularities) and how to glue them, as well as to how one extracts computable data from the algebras. In the end this approach gives us our target space and it's Betti numbers, as well as the category of D-branes on the space. A full description of the homological and algebraic tools used in the paper may be found in refs. [10, 21, 23] and [25]–[31].

The paper is organized as follows: in section 2 we begin with an introduction to non-commutative algebraic geometry in the setting of [11]. In particular we define what a non-commutative point is (a point-like brane) and how to build the target space geometry from these objects.

In section 3 we present the principal algebraic construction we study, and define the *crossed product* of an algebra (which can be taken to be commutative) with a discrete group of automorphisms. This is how one constructs an orbifold space as a non-commutative geometry. We consider in the following sections a number of examples, primarily the well-known $\mathbb{C}^2/\mathbb{Z}_n$ orbifold and associated orbifolds with discrete torsion. Using the crossed product we show that unorbifolding an Abelian orbifold gives an algebra which is (locally) Morita equivalent to the original commutative algebra. In particular this means that the K-theory of both algebras is the same, but more is true, their D-brane categories are the same as well; so both algebras are describing the same geometry.

In section 4 we exploit ideas presented in [32] to understand an orbifold of an orbifold; that is, we consider the problem of orbifolding a general non-commutative space, and in particular we focus on the problem of how one distinguishes ungauged symmetries from gauged symmetries. This is not an issue for a commutative algebra, but it is a subtle technical point which might provide anomalies to construct backgrounds with certain geometric data in string theory, as given for example in [33].

In section 5, we consider other non-orbifold singularities, such as the conifold geometry and it's brane realization [34]. We discuss what algebra should be associated with the conifold. As well we recall the conifold of [18] and it's D-brane realization [35]. This demonstrates that two distinct non-commutative geometries resolve the same geometric singularity, and they are not Morita equivalent to each other, as the spectrum of fractional branes at the singularity is different. This demonstrates that these two conifolds are topologically distinct.

Next, in section 6 we consider the problem of calculating the massless open string spectrum between two distinct D-branes, based on ideas of Douglas [20] generalized to non-commutative geometry. Here we find our definition of regularity: namely that any D-brane locally can be associated to the decay product of a finite number of (locally) space-filling branes and anti-branes (that is, any coherent sheaf admits locally a projective resolution). The number of strings between two branes is topological and in this setting corresponds to calculating the derived functor for the module morphisms $hom(A, B)$, namely the $\mathcal{E}xt$ groups. Here is where we first find a computation in homological algebra and where we see a clear connection to [19, 20]. In particular, the regularity condition guarantees that the process of calculating these intersection numbers terminates. We show in the example $\mathbb{C}^2/\mathbb{Z}_n$ how the quiver diagram construction of [22] is exactly the computation of the groups $\mathcal{E}xt^1(A, B)$ for A, B fractional branes (which are the nodes of the quiver diagram).

Following this construction, we focus our attention on closed strings in section 7. We argue that the proper tool to understand topological closed string states is the cyclic homology of the algebra. The reasons behind this proposal rest on the map between closed string states and gauge invariant operators in the AdS/CFT correspondence, which identify closed string states as traces. Cyclic homology is a homology theory of generalized traces and it is an invariant under Morita equivalence. Cyclic homology is dual to K-theory, as one can produce a pairing between them. Thus it is the natural receptacle for closed string states as it allows us to compute topological couplings between D-branes and cohomology classes (RR topological fields). From here we use our results to compute the Betti numbers of the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with and without discrete torsion, and we show how these are related to the topology of the fibration of the non-commutative geometry over the singular set. Finally we conclude with open problems and possible future lines of development.

2. Algebraic preliminaries

D-brane configurations may be given by systems of equations, which may be thought of as the moduli space problem in the low energy field theory limit. More generally, if one considers collections of D-branes and anti-D-branes, one still gets a system of equations which are derived from the effective action of the tachyon condensate plus the massless fields (everything else can be integrated out). In our approach these tachyons will be implicit, and will appear later when we study the homological functors. Our approach applies to topological branes, in the sense that we are ignoring the metric (and also the B-field which is just the complexification of the metric). The latter is important for calculating Chern classes, which determine cohomology rather than K-theory classes, but may otherwise be ignored. Thus, we can focus on holomorphic data, and deal with the algebraic geometry of non-commutative algebras.

Our approach is based on geometric realizations of string theories. Non-geometric phases of string theory are not described in this framework. It is also a low energy approach, and assumes that the non-commutative algebras we begin with are a good description of the space. With this philosophy in mind we sacrifice exact closed string backgrounds such as those built out of RCFTs as well as their boundary states (see [36, 37, 38] for very interesting work in those directions). However, our construction can describe generic points in the moduli space of complex structures, for which no known exact conformal field theory description exists. A good review of the ideas necessary to understand D-branes at generic points can be found in [39]. Further, we restrict ourselves to special classes of algebras, which in some sense are almost commutative, which is exactly the opposite limit that one considers when one studies deformation quantization (as in the work [40, 41]). In principle our results should admit generalizations in this direction, but the algebraic tools will probably change drastically. The algebras that we consider here are finitely generated[2] over their (local) center; that is, we require the non-commutative algebra to be a sheaf of algebras over a possibly singular commmutative space. As we will mainly be interested in singular examples to understand how the non-commutative geometry resolves the space, we will assume that we are in a singular manifold already.

This choice of algebras seems rather restrictive. However, any global orbifold space by a finite group action admits a description in terms of these algebras, see section 3, and thus any space which has local singularities of the orbifold type also admits this kind of description. In general it is probably useful to consider a more general space defined by an aritrary $\mathbb{C}^*$ algebra as a target space for D-branes. However we might lose the commutative algebraic geometry tools that are so useful when dealing with computations in Calabi Yau spaces. Some of the tools we have described carry over, but the tools required to make these spaces tractable will look very different than the presentation we are giving here [23], and moreover the holomorphic structure of the space might be lost.

The above description we have given is essentially an Azumaya algebra over the singular complex manifold, and these have already appeared in [43, 44] as describing branes on quotient spaces with some B-field torsion, as well as branes on tori. However, our description is local and not based on global quotients.

A simple example is that of D-branes at an orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ with discrete torsion [35, 45]. Here we get a $\mathrm{U}(N)$ gauge theory with three adjoints $\phi_{1,2,3}$ and superpotential

$$W = \mathrm{tr}(\phi_1\phi_2\phi_3 - q\phi_2\phi_1\phi_3) \tag{2.1}$$

with $q^n = 1$. The F-term constraints are then seen to be given by

$$\phi_1\phi_2 - q\phi_2\phi_1 = 0 \tag{2.2}$$

and it's cyclic permutations, which are holomorphic constraints giving rise to a non-commutative algebra.

[2] It is probably better to work with the classes of algebras which are locally Morita equivalent to one of these, but this requires a more elaborate structure [42, 30].

The above algebra is a non-commutative algebra which encodes the target space. In [11] this theory was analyzed using non-commutative geometry tools. Points in the (non-commutative) space were defined as irreducible representations of the above algebra. By Schurs lemma, any element of the center evaluated on a representation will be proportional to the identity, and this value will give a point in the commutative space of the center of the algebra. The center of the algebra thus gives exactly the ring of holomorphic functions of the orbifold, what one would naturally call the orbifold space: the target space for the closed strings. Thus there are two geometries: one non-commutative associated to the point-like D-branes, and a commutative one associated to the closed strings, in a spirit much like the work of Seiberg and Witten[6]. The non-commutative geometry is fibered over the commutative space, and most of the geometric calculations we will do, as well as our intuition, depend crucially on understanding this fibration in detail. More details as to how this construction should be introduced can be found in [11].

A generic D-brane will be a coherent sheaf of modules over the local algebras which are finitely presented, and as such we are naturally in the topological B-model. We will assume that we are in a large volume phase of the Calabi Yau manifold (except for the singularities), so that we are in a geometric phase of the string field theory (and not a Landau Ginzburg phase for example), but in general we will ignore the issue of the metric all together and we will just be interested in the questions that depend only on the holomorphic structure of the space.

On the algebra defined by the relations (2.2) it is natural to construct the closed string spectrum associated to the singularity by using the AdS/CFT correspondence. The one particle closed string states are then the single trace elements of the chiral ring of the above conformal field theory. The support of a closed string state is the set of non-commutative points where evaluation of the trace gives a non-zero result. It was found in [11] that this chiral ring was precisely the Hoschild homology in degree zero of the above algebra $HH_0(\mathcal{A})$. This in turn is the same group as the cyclic homology group $HC_0(\mathcal{A})$, see [27] for more mathematical details, and it is suggestive to build a theory of closed string states by using these groups.

The data given in general will be the local (on some patch) descriptions of the non-commutative algebras as a set of generators and relations. Given these, we need to calculate the center of the (local) algebra and its representation theory. This procedure will give us a model for our (local) target space. The prescription needs to be supplemented by a recipe for gluing algebraic data (localization theory) and extracting global invariants (global holomorphic sections, betti numbers, moduli spaces, etc) of the space. The localization takes place by taking denominators in the center of the algebra, namely, we glue the 'geometry' of the closed strings. For the global invariants we will only develop the intersection theory of branes and the calculation of betti numbers on orbifold Calabi Yau spaces.

3. Crossed products

The low energy degrees of freedom on supersymmetric point-like branes at orbifold singularities are constructed by taking images of branes and projecting onto gauge invariant states. This construction was carried out in detail in [22] and gives rise to field theories which are represented by a quiver diagram.

The construction exploits the fact that the group action on the orbifold acts via a gauge transformation on the brane, and the low energy degrees of freedom are encoded in the projection condition

$$\gamma(g)\phi^i\gamma(g)^{-1} = R^i_j\phi^j\,, \tag{3.1}$$

where the ϕ^i are directions normal to the singularity, and R^i_j is the representation under which they transform. The $\gamma(g)$ are given by the regular representation of the group (possibly twisted by some cocycle to account for discrete torsion [35, 45]). The low energy degrees of freedom are captured by the ϕ^i which satisfy the above conditions, and the superpotential is derived from the superpotential of the parent $N = 4$ theory.

We want to read 3.1 as an algebraic identity in the non-commuttaive space. First, we have the ϕ^i which commute among themselves (since they come from the unorbifolded theory whose F-terms are $[\phi^i, \phi^j] = 0$); these form an algebra $\mathcal{A} \sim \mathbb{C}^3$. Next, according to (3.1), we need to add generators to the algebra for each g, which we will call e_g. These are such that $e_g \cdot e_{g'} = \epsilon(g, g')e_{gg'}$ with ϵ the twisting cocyle. The twisting cocycle is how one introduces discrete torsion[35, 45].

In this algebra the outer automorphism of the commutative algebra $[\phi^i, \phi^j] = 0$ given by the orbifold group action is gauged, and becomes an inner automorphism of the non-commutative algebra, that is, conjugation by the elements of G becomes part of the algebra. This new larger algebra is called the *crossed product algebra* (see [10] for details) of $\mathcal{A}$ and ΓG (the group algebra), and we will denote it by $\mathcal{A}\boxtimes G$. This recipe can be placed in a very general context: given an automorphism of a non-commutative algebra $\mathcal{A}$, and for G discrete the algebra $\mathcal{A}\boxtimes G$ has the same properties as stated above.

This algebra $\mathcal{A} \boxtimes G$ has various important properties. First, the center of the new algebra is the ring of invariants of $\mathcal{A}$ under the action of G, and thus the center of the algebra captures the commutative geometric quotient. In [11, 46, 47] it was argued that the center of the algebra is a natural candidate for the space where closed strings propagate. The non-commutative geometry is associated to D-branes, and the commutative geometry of the center is associated to closed strings, in a spirit that follows [6].

Secondly, the K-theory of this algebra corresponds to the (twisted) G equivariant K-theory of the algebra $\mathcal{A}$ [10], and thus the topological classes of D-branes are captured automatically, as they correspond to K-theory classes [7, 9] (See also ref. [48], where an attempt was made to make this equivariant construction precise by using commutative geometry alone.) The formalism thus allows for a computation of the K-theory classes which are supported at singularities. These will be local quivers for each singularity.

3.1 A first example: $\mathbb{C}^2/\mathbb{Z}_n$

Probably the best understood orbifold in string theory is $\mathbb{C}^2/\mathbb{Z}_n$. We will show how to recover the quiver diagram for the above just by studying the crossed product algebra. Thus, consider two complex generators x, y for $\mathbb{C}^2$ which commute with each other. The crossed product algebra will be generated by x, y, σ, where σ is the non-trivial generator of $\mathbb{Z}_n$ which satisfies

$$\begin{aligned} \sigma x \sigma^{-1} &= \omega x \\ \sigma y \sigma^{-1} &= \omega^{-1} y \\ \sigma^n &= 1 \end{aligned} \tag{3.2}$$

with $\omega^n = 1$. Consider first the center of the algebra above. Any element of the algebra can be written in the form

$$\sum_{i=0}^{n-1} \pi_i(x, y)\sigma^i\,. \tag{3.3}$$

The elements of the center are obtained as follows: requiring that this commutes with x, y forces us to consider only those which are of the form $P(x, y)$ with no dependence on σ. By commuting $P(x, y)$ with σ we see that the monomials of $P(x, y)$ must be such that they are invariant under the action of the group, so that we recover the ring of invariants. The center of the algebra is generated by $u = x^n, v = y^n, z = xy$. These together satisfy the constraint $uv = z^n$ which identifies the algebra as the $\mathbb{C}^2/\mathbb{Z}_n$ orbifold.

Given the non-commutative algebra, it can be seen that the non-commutative points are fibered over the algebraic geometry of the center [11], and given a point of this commutative geometry it is necessary to understand this fibration.

Now, we will build the irreducible representations of the algebra for a regular point (a point which is not the fixed point). This will correspond to having specific values of (u, v). For each, there will be a unique irreducible representation, which acts on a vector space with basis $|0\rangle, \dots, |n-1\rangle$. On this basis, we have $\sigma|k\rangle = \omega^k|k\rangle$, and

$$\sigma = \mathrm{diag}(1, \omega, \omega^2, \dots)\,, \qquad x = \alpha Q\,, \qquad y = \beta Q^{-1} \tag{3.4}$$

with Q a shift operator on the basis $Q|k\rangle = |k + 1 \mod (n)\rangle$. This is exactly the regular representation of this group, which we will call $R(\alpha, \beta)$. The representation space on which the algebra is acting is a left module of the algebra $\mathbb{C}^2/\mathbb{Z}_n$. (α, β) refer to coordinates in $\mathbb{C}^2$.

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Notice that the irreducible representations $R(\alpha, \beta)$, $R(\omega\alpha, \omega^{-1}\beta)$ are related to each other by conjugation by σ. Thus the parameter space for α, β is also $\mathbb{C}^2/\mathbb{Z}_n$. This is a bulk point-like D-brane. The algebra of a bulk point (or a small neighborhood of it) is the algebra of $n \times n$ matrices tensored with the local commutative algebra of the point, $\mathbb{C}(\alpha - \alpha_0, \beta - \beta_0) \otimes M_n$, and this is locally Morita-equivalent to a standard commutative geometry. This is depicted in figure 1.

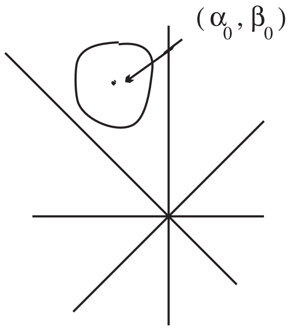

Figure 1: Small coordinate patch around α_0, β_0.

The only singularity in the commutative space happens when we take $\alpha, \beta \to 0$. The representation theory of the non-commutative algebra becomes reducible at that point, and we obtain n distinct irreducible representations labeled by k, namely the one dimensional spaces $|k\rangle$ where σ takes all of its possible different characters. These n irreducible representations of $\mathbb{Z}_n$ correspond to the brane fractionation at the singularity, and they are the wrapped branes on the singular cycles [49, 50], (these are the new equivariant compact K-theory classes).

We can now draw a quiver for fractionation. For each irreducible representation of the algebra $\mathcal{A} \boxtimes G$ at the singularity we write a node. In the above example we get n nodes, as we have n different irreducibles at $x = y = 0$.

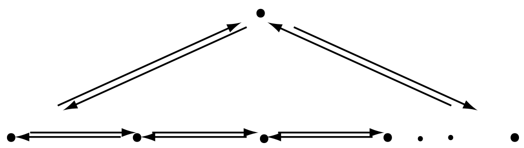

Figure 2: Quiver diagram for the A_{n-1} singularity.

Now, these are obtained by taking α, $\beta \to 0$. The matrices of x, y are off-diagonal in a precise fashion when we take σ to be diagonal (they are given by the shift operator), and for each non-zero entry of each of the x, y matrices we write one arrow between the nodes (each node corresponds to a diagonal entry in the matrix) that it relates by matrix multiplication, and it is giving us a massless field on the field theory of the D-brane. Later we will redo this calculation in a more formal setting in section 6 where the above will be made precise. The resulting quiver is given in figure 2

The n irreducible representations associated to the fractional branes correspond to the irreducible representations of the fixed point group algebra (this is the group that leaves the fixed point invariant), and these are the irreducible representations of $\mathbb{Z}_n$. Thus the nodes are naturally associated with irreducible representations of $\mathbb{Z}_n$, as given by the quiver construction in [22]. These in turn are associated each with a projector in the group algebra of $\mathbb{Z}_n$ (a K-theory class in the group algebra of $\mathbb{Z}_n$).

The chiral fields corresponding to the individual arrows in the diagram are then obtained by taking $\phi^1_{k,k+1} = \pi_k x = x\pi_{k+1}$, and $\phi^2_{k+1,k} = \pi_{k+1} y = y\pi_k$, with the π_i the n projectors in the group algebra. These are the independent fields on the D-brane.

It is easy to show that the identities that the ϕ^i_{jk} satisfy are exactly the ones that correspond to the equations of motion derived from the superpotential of the quiver diagram. The projectors are part of the algebra and they indicate on which of the nodes the chiral multiplets begin. These are also discrete variables that are available in the quiver diagram. Actually, the n projectors are given by the character formula $\pi_i = (1/n)\sum_i \omega^{ki}\sigma^i$, so one can reconstruct the group algebra by taking linear combinations of the projectors. Thus in this case the algebra of massless fields is exactly the crossed product algebra and not just a subalgebra as hinted in [51].

We have ignored so far the presence of D-terms. It is known that taking the D-terms away from zero resolves the singularity [52], and in the representation theory one finds that in the full $\mathbb{C}^*$ algebra the branes do not fractionate once this is done, but then one is also forced to consider patches of algebras to account for the algebraic geometry of the exceptional divisor. In principle the holomorphic data of both situations is the same.

3.2 Unorbifolding the orbifold

Orbifolds by abelian groups can be undone in string theory. This requires using the quantum symmetry of the orbifold to undo the original twisting, and the new twisted sectors will recover what was lost originally. Here, we will show that this operation is available in the crossed product construction. There are two parts to this construction. We will first analyze the previous example $\mathbb{C}^2/\mathbb{Z}_n$, and show how the unorbifold recovers the original space. Then we will comment on the general case.

The quantum symmetry on the $\mathbb{C}^2/\mathbb{Z}_n$ acts by phases on the twisted sectors. Twisted sectors can only couple to the fractional branes, and hence, the quantum symmetry exchanges the n irreducible representations of $\mathbb{Z}_n$. It must also act as an automorphism of the algebra, and it is seen that the generator of the symmetry acts by sending $\sigma \to \omega\sigma$, while it keeps the original algebra of $\mathbb{C}^2$ unchanged. This is just as well, as the quantum symmetry is not a symmetry of the original algebra. In the same spirit as before, we will implement the orbifold by the quantum symmetry $(\mathbb{C}^2/\mathbb{Z}_n)/\mathbb{Z}_{nq}$ by introducing a new generator $\tilde{\sigma}$, such that

$$\tilde{\sigma}\sigma\tilde{\sigma}^{-1} = \omega\sigma \tag{3.5}$$

and $\tilde{\sigma}^n = 1$.

The center of the new algebra $(\mathbb{C}^2/\mathbb{Z}_n)/\mathbb{Z}_{nq}$ is generated by

$$\tilde{y} = y\tilde{\sigma}^{-1}, \quad \tilde{x} = x\tilde{\sigma}\,. \tag{3.6}$$

Notice that these two are unconstrained, thus they represent the geometry of $\mathbb{C}^2$. Changing variables to $\tilde{y}, \tilde{x}, \sigma, \tilde{\sigma}$ we get that the algebra is a tensor product of algebras $(\mathcal{A}/\mathbb{Z}_n)/\mathbb{Z}_{nq} \sim \mathcal{A}(\tilde{x}, \tilde{y}) \otimes A(\sigma, \tilde{\sigma})$

The factored algebra $A(\sigma, \tilde{\sigma})$ has a unique irreducible representation and as an algebra it is isomorphic to the set of $n \times n$ matrices. Thus the algebras of $(\mathcal{A}/\mathbb{Z}_n)/\mathbb{Z}_{nq}$ and $\mathbb{C}^2$ are related by Morita equivalence.

Notice that the extra matrices that we have obtained are discrete degrees of freedom and they do not contribute new branches to the moduli space — they don't contribute degrees of freedom that are light, and the effective actions of the light degrees of freedom for the two algebras agree. Thus both algebras represent the same low energy physical system.

For a general abelian orbifold of a commutative space one can do the same construction: each of the generators y_i of the algebra can be chosen to transform in a specific irreducible representation associated to a character χ_i of G. The quantum symmetry will act via the dual group $\hat{G}$, the group of characters of G. Thus we will have generators associated to each character of g which will satisfy

$$\chi \cdot g \cdot \chi^{-1} = \chi(g) g \,, \tag{3.7}$$

where $\chi(g)$ are the evaluation of the character χ in the element g, and these are just numbers.

One can define new variables $\tilde{y}_i = y_i \chi_i$, and these are constructed so that they commute with the elements of $G, \hat{G}$. Any relation which the y_i satisfy is also satisfied by the $\tilde{y}_i$, a consequence of the fact that G acts by an automorphism of the algebra of the y_i. Again we find a tensor product factorization where we obtain the original algebra and the algebra generated by the g, χ. The latter algebra has a unique irreducible representation and is thus isomorphic to the set of $|G| \times |G|$ matrices. Again the unorbifold and the original algebra differ only by the tensor product with $|G| \times |G|$ matrices and are Morita equivalent. The moduli spaces of vacua of the two agree, and the geometry associated to the two is the same.

Notice that in the example above the fixed point set of the action of the quantum symmetry is everything but the singular set. In this sense in the commutative algebra of the center the quantum symmetry is a non-geometric symmetry. In the non-commutative geometry the action is geometrical and it exchanges the two fractional points. These combine into a single irreducible, while the fixed points split in two, just the opposite process of the original orbifold which combined sets of two points into one. This is an example of Takai duality, (see [10, theorem 10.5.2]), this fact was pointed out in [51] without example.

4. Orbifold of an Orbifold and discrete torsion

In this section we will develop ideas presented in [32], where it was shown that one could exploit sequences of groups $H \to G \to G/H$ to get quiver diagrams for various orbifolds, by taking the quotient stepwise

$$\mathcal{M}/G \sim (\mathcal{M}/H)/(G/H)\,. \tag{4.1}$$

We will do this exercise to point out some of the pitfalls one might encounter in doing orbifolds of more complicated algebras, while we know what the final product should be. A discrete group will act on the algebra as an algebra automorphism, and thus it will be a subgroup of $Aut(\mathcal{A})$. Of these automorphisms there are some which are trivial, as they are constructed by conjugation by an invertible element of the algebra. These do nothing to the irreducible representations of the algebra, and they have no geometric action on the non-commutative moduli space. Moreover, the representation theory of an algebra is well defined only up to conjugation, and this conjugation is exactly the gauge symmetry on the D-branes [11]. These are inner automorphisms, corresponding to gauge transformations and should not be considered.

The group of inner automorphisms, $Inn(\mathcal{A})$, is a normal subgroup of the group of automorphisms of the algebra. As such, we can take the quotient group and identify

$$Out(\mathcal{A}) = Aut(\mathcal{A})/Inn(\mathcal{A})\,, \tag{4.2}$$

the group of outer automorphisms of the algebra, which is the set of symmetries which are not gauged. Thus we should be modding out only by elements of $Out(\mathcal{A})$.

In the case where $\mathcal{A}$ is a commutative algebra, the inner automorphisms are trivial so we get that $Out(\mathcal{A}) = Aut(\mathcal{A})$. However, the outer automorphism on a non-commutative algebra $\mathcal{A}$ is only well defined in $Aut(\mathcal{A})$ up to conjugation by an invertible element, thus when we write the physical quotient we want, we need to take this into account to get the new orbifold algebra right. This is a lifting problem from $Out(\mathcal{A})$ to $Aut(\mathcal{A})$. In the example in section 3.2 the lifting was such that each element of $Out(\mathcal{A})$ was represented by an unique element in $Aut(\mathcal{A})$, thus the lifting problem there was trivial.

As a first step, we will consider examples where this is not an issue, and then the crossed product construction will give all of the information of the new algebra. Consider the orbifold $\mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$, where the group acts by

$$\begin{aligned} e_1 &: (x,y,z) \to (\omega x, y, \omega^{-1} z) \\ e_2 &: (x,y,z) \to (x, \omega y, \omega^{-1} z) \end{aligned} \tag{4.3}$$

with $\omega^n = 1$ and we want to take the orbifold stepwise as $(\mathbb{C}^3/\mathbb{Z}_n)/\mathbb{Z}_n$.

At the first step, we introduce the generator e_1 which is such that

$$e_1 x e_1^{-1} = \omega x\,, \qquad e_1 y e_1^{-1} = y\,, \qquad e_1 z e_1^{-1} = \omega^{-1} z \tag{4.4}$$

and $e_1^n = 1$. Because of the orbifold, we now have a quantum symmetry that acts on the generator of the group by multiplication by a character $e_1 \to \chi(e_1) e_1$. This quantum symmetry is also $\mathbb{Z}_n$, thus we have a group $\mathbb{Z}_n \times \mathbb{Z}_n$ of automorphisms of the algebra– one $\mathbb{Z}_n$ coming from the geometric realization of the second $\mathbb{Z}_n$ on the original space, and another $\mathbb{Z}_n$ from the quantum symmetry. When we do the second $\mathbb{Z}_n$ geometric orbifold, we can choose to twist it by an element of the quantum symmetry.

Thus at the second step we introduce a generator e_2 and we can have

$$e_2 e_1 e_2^{-1} = e_1 \chi(e_1)\,. \tag{4.5}$$

Here, we have n choices for $\chi(e_1) = \omega^\alpha$. This choice in the orbifold is the prototypical example of a discrete torsion phase for the group $\mathbb{Z}_n \times \mathbb{Z}_n$. For a general group G, the phase will be an element of $H^2(G, \mathrm{U}(1))$, and each such choice represents a twisted group multiplication law.

Because of the twist by the quantum symmetry, the second $\mathbb{Z}_n$ will act on the fractional branes and permute them, thus it will identify nodes in the quiver diagram and the new quiver will be different. This twist is also responsible for having monodromies of the fractional branes around the codimension 3 singularities. We will refer the reader to [47, 32] for more details.

Now let us specialize to the case where $\chi(e_1)$ is a primitive n-th root of unity, so we are in the situation with maximal discrete torsion. This was the example which was used to develop the non-commutative moduli space theory in [11]. There the algebra of the three superfields was found to satisfy

$$[\phi_i, \phi_{i+1}]_q = 0\,, \tag{4.6}$$

where $[A, B]_q = AB - qBA$ with $q^n = 1$.

In our case the algebra is $[x, y] = [x, z] = [z, y] = 0$ plus the relations derived from the group algebra, and the two algebras look very dissimilar. As e_2 and e_1 no longer commute, it is useful to change variables (in a way similar to (3.6)) to $\tilde{x}, \tilde{y}, \tilde{z}$, such that e_2, e_1 commute with the new variables. If this can be done, the algebra factorizes.

The appropriate change of variables is the following:

$$\tilde{x} = x e_2^{-k}\,, \qquad \tilde{y} = y e_1^k\,, \qquad \tilde{z} = z e_1^{-k} e_2^k\,, \tag{4.7}$$

the value of k chosen so that $\omega \chi(e_1)^k = 1$. As a result, the algebra is a tensor product

$$\mathcal{A}/(\mathbb{Z}_n \times \mathbb{Z}_n)_{d.t.} = \mathcal{A}[\tilde{x}, \tilde{y}, \tilde{z}] \otimes \mathcal{A}[e_1, e_2]\,. \tag{4.8}$$

The algebra $\mathcal{A}[\tilde{x}, \tilde{y}, \tilde{z}]$ is non-commutative, and it is simple to show that there are relations such as

$$\tilde{x}\tilde{y} = \omega^{-k} \tilde{y}\tilde{x}\,. \tag{4.9}$$

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This can also be written $[\tilde{x}, \tilde{y}]_q = 0$ with $q = \omega^{-k}$, which is also an n-th root of unity. The group algebra generated by $\mathcal{A}[e_1, e_2]$ with the maximal twisting has only one irreducible representation, and it is isomorphic as an algebra to the set of $n \times n$ matrices. Thus the two descriptions of the algebra $\mathcal{A}/(\mathbb{Z}_n \times \mathbb{Z}_n)_{d.t.}$ that we have given are Morita equivalent.

In the case of more general nonabelian orbifolds, this is also true although it is much harder to show. Thus the algebra of light fields on a regular D-brane at an orbifold singularity is Morita equivalent to the algebra of the crossed product algebra, which represents the quotient. In particular, Morita equivalence implies that the K-theory of both algebras is the same. In different situations one representation might be more useful than the other one, but the physical content of both is the same.

4.1 Lifting issues

4.1.1 $\mathbb{C}_2/\mathbb{Z}_4$

Consider now the sequence $\mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2$. We will use this sequence to calculate the orbifold $\mathbb{C}^2/\mathbb{Z}_4$, by going through the long route $(\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_2$. This will serve to illustrate a few points on the lifting issues from $Out(\mathcal{A})$ to $Aut(\mathcal{A})$. The first step in the process has already been discussed in section 3. The second step is the second $\mathbb{Z}_2$ automorphism of this algebra.

We will take the automorphism to be given by the expected geometric action

$$e_2 : (x, y, \sigma) \to (ix, -iy, \sigma)\,. \tag{4.10}$$

At first sight it looks as if e_2 is an outer automorphism of order four, and hence we are not dividing by a $\mathbb{Z}_2$, but we must note that

$$e_2^2 : (x, y, \sigma) \to (-x, -y, \sigma) \tag{4.11}$$

which is identical to conjugation byσ. The automorphism is of order 4 in $Aut(\mathbb{C}^2/\mathbb{Z}_2)$, but it is of order 2 in $Out(\mathbb{C}^2/\mathbb{Z}_2)$. To obtain the proper orbifold, we do the crossed product as before, but we need to introduce the extra relation

$$e_2^2 = \pm\sigma \tag{4.12}$$

(the choice of sign is inconsequential.) That is, we equate the trivial outer automorphisms with an appropriate inner automorphism. The resulting algebra is indeed that of the $\mathbb{C}^2/\mathbb{Z}_4$ orbifold.

Thus given a subgroup of $Out(\mathcal{A})$, we want to lift to a subgroup of $Aut(\mathcal{A})$, and as we have seen, this might not always be trivial. If we consider a generating set of group elements for $Out(\mathcal{A})$, we choose one lift in $Aut(\mathcal{A})$, and we might obtain a larger group. We take the crossed product of this new group in $Aut(\mathcal{A})$ with $\mathcal{A}$, and thus obtain a space which is orbifolded too much, in the sense that we are gauging again transformations which were already gauged. To repair this, we must find extra relations (as in (4.12)) needed to remove the overcounting of inner automorphisms. Finding such relations can be a non-trivial task. To see this, let us consider one such example with a non-abelian orbifold.

4.1.2 $\mathbb{C}^2/\hat{E}_8$

The orbifolds of $\mathbb{C}^2/\Gamma$ which are Calabi-Yau are classified by the ADE groups. Consider $\mathbb{C}^2/\hat{E}_8$, where $\hat{E}_8$ is the discrete subgroup of SU(2) corresponding to the icosahedron group. The group $\hat{E}_8$ fits into an exact sequence

$$\mathbb{Z}_2 \to \hat{E}_8 \to E_8 \,, \tag{4.13}$$

where E_8 is the image of $\hat{E}_8$ in $SO(3)$. Thus we can consider

$$\mathbb{C}^2/\hat{E}_8 \simeq (\mathbb{C}^2/\mathbb{Z}_2)/E_8 \,. \tag{4.14}$$

The lift of E_8 to $\hat{E}_8$ is non-trivial, as it corresponds to a non-trivial central extension of E_8. In particular, this extension is associated to the possibility of discrete torsion in E_8 [53].

Obviously we know what final algebra we want, as it is given by the crossed product of the algebra of $\mathbb{C}^2$ and $\hat{E}_8$. One can easily identify the lift of E_8 to $\hat{E}_8$. Although the action of E_8 on the algebra $\mathbb{C}^2/\mathbb{Z}_2$ is linear in the coordinates, the coordinates do not transform as any irreducible representation of the E_8 algebra. Indeed, only the elements of the center of $\mathbb{C}^2/\mathbb{Z}_2$ (which are gauge invariant variables) transform in this manner.

The E_8 leaves the fractional brane representations at the fixed point of $\mathbb{C}^2/\mathbb{Z}_2$ fixed, and on one of them it acts as the algebra of E_8, while in the other one the action is twisted by the discrete torsion cocycle. Thus the action of E_8 on the different fractional branes can act with or without discrete torsion, and when one assembles the quiver diagram one gets both possible types of representations. The quiver is drawn in figure 3.

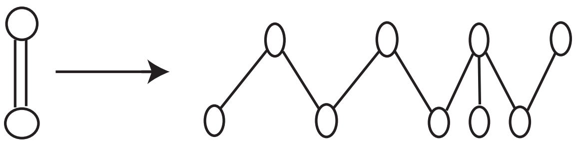

Figure 3: Quiver diagram of the $\hat{E}_8$ singularity drawn using the two step technique of [32].

In other cases in the literature the possibility of discrete torsion gave disconnected diagrams [54]. This is because the coordintaes transformed as representations of the group G, not just of the covering group. It has been argued that all choices of discrete torsion can appear on the same footing [55, 56], based on a T-duality between the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with and without discrete torsion. Here we show an example which involves only one non-commutative geometry without changing between 'mirror' geometries.

4.1.3 $\mathbb{C}^3/\mathbb{Z}_4$

A second example we will consider is the $\mathbb{C}^3/\mathbb{Z}_4$ orbifold, which has been addressed by Joyce [33], who argued that it should admit distinct resolutions. The orbifold acts by

$$(x, y, z) \to (ix, iy, -z)\,. \tag{4.15}$$

We will again consider the same two step procedure discussed previously

$$\mathbb{C}^3/\mathbb{Z}_4 \simeq (\mathbb{C}^3/\mathbb{Z}_2)/\mathbb{Z}_2\,. \tag{4.16}$$

However, we will twist the second $\mathbb{Z}_2$ action by the quantum symmetry of the first $\mathbb{Z}_2$ (as we are instructed to do by [33]). In string theory this is apparently an allowed process, but there is a question as to how this is done for D-branes using methods of non-commutative geometry.

The first step will give us the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ space, which has generators x, y, σ, z. Now, we want the second $\mathbb{Z}_2$ group to act by

$$e_2 : (x, y, \sigma, z) \to (ix, iy, -\sigma, z)\,. \tag{4.17}$$

From this, we obtain that in the crossed product we should have the relation

$$e_2 \sigma e_2^{-1} = -\sigma \tag{4.18}$$

and one can also see that e_2^2 corresponds to conjugation by σ. However, we cannot set $e_2^2 = \pm\sigma$, as this is incompatible with (4.18). This obstruction happens because on a non-commutative geometry the inner automorphisms act by conjugation, and this means that it is an element of $\mathrm{GL}(1, \mathcal{A})/\,\mathrm{U}(1, \mathcal{Z}\mathcal{A})$. That is, the elements of $Inn(\mathcal{A})$ are invertible elements in $\mathcal{A}$ modded out by their phase which can possibly vary continuously with the parameters of the center. Thus there is a possible ambiguity in the lift of e_2^2 to an element of $\mathrm{GL}(1, \mathcal{A})$ in order to get the relations in the algebra right. As we see, this is a non-trivial problem. If we orbifold by e_2 as it stands, it is an automorphism of order four as we cannot find the new relations in the algebra: we are orbifolding too much and we do not recover Joyce's orbifold.

It is possible that there is some Morita equivalent construction (obtained by tensoring in matrices) that gets around this problem. If this cannot be done, then it represents a stringy obstruction to Joyce's proposal.

5. Non-orbifold singularities

So far we have seen how non-commutative geometry describes orbifold singularities and D-bane fractionation at singularities. In this section we will show that non-commutative geometries can also be used to describe other singularities which are not of the orbifold type. As an example, consider the conifold singularity of Klebanov and Witten [34]. This singularity is obtained by a deformation of the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ space.

The algebra generators are σ, x, y, z, with non trivial relations given by $\sigma^2 = 1, \sigma x\sigma = -x, \sigma y\sigma = -y, \sigma z\sigma = z$. This is enough to give us the right quiver diagram of the singularity $\mathbb{C}^2/\mathbb{Z}_2$. This fixes the commutation relations with the discrete variable σ.

The theory has a modified superpotential, which is given by

$$W = \mathrm{tr}(xyz - yzx) + \frac{m}{2}\mathrm{tr}\sigma z^2 \tag{5.1}$$

with the presence of σ in the last term telling us that we have turned on a twisted sector of the orbifold (we will make this clear later in section 7) and from here we get the additional algebraic relations

$$\begin{aligned} xy - yx &= -m\sigma z \\ yz - zy &= 0 \\ xz - zx &= 0\,. \end{aligned} \tag{5.2}$$

From these relations we see that z is in the center of the algebra, and moreover it is redundant as a generator as we have $z = m^{-1}\sigma[y, x]$. The choice of including or not including z is equivalent to integrating it out in the effective field theory. Also notice that $u = x^2$ and $v = y^2$ commute with σ, and it is easy to prove that they are part of the center of the algebra. If $m = 0$, xy is in the center of the algebra; when $m \neq 0$, this is deformed to $w = xy + m\sigma z/2$ (equivalently, we can write this as $(xy + yx)/2$).

The relations between u, v, z, w are given by

$$uv = w^2 - m^2z^2/4 \tag{5.3}$$

and this geometry is the geometry of a conifold $uv = u'v'$ after a change of variables.

Here, the slices at constant z are deformations of the $\mathbb{C}^2/\mathbb{Z}_2$ which give a smooth ALE space. The only singularity of the conifold is at codimension 3 and it occurs at $u = v = w = z = 0$.

To solve for the irreducible representations of the algebra we diagonalize σ and the elements of the center, and then x, y are seen to be off-diagonal in this basis. Away from $u = 0$ we can choose $\sigma = \sigma_3$ (the Pauli matrix), and $x = \alpha\sigma_1$. Then we can take $y = \alpha'\sigma_1 + \beta\sigma_2$, where $\alpha\beta = imz/2$ and

$$w = \alpha\alpha' I \tag{5.4}$$

which as an element of the center is appropriately proportional to the identity. Thus the representation is labeled by three complex numbers, say α', α, z. It is straightforward to show that these are all of the irreducible representations.

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One can show that one can cover the full conifold by such patches of representations where $u \neq 0, v \neq 0, w \neq 0$ respectively, and that for each of those there is a unique irreducible representation. Thus locally the space is Morita equivalent to a commutative manifold away from the singularity. The only point which is left to discuss is $u = 0, v = 0, w = 0$ which is the singular point. There, the generic irreducible representation can be constructed by the limit $z \to 0, \alpha \to 0, \alpha' \to 0$ in the above parameterization. In this limit, the representaton becomes reducible. Thus in this conifold the brane splits in two when it reaches the conifold singularity. One can recover the quiver diagram of the conifold from this splitting, and we get back the original quiver diagram of the $N = 2$ theory which was deformed. Here the conifold is 'resolved' by fractional branes. Thus we have extra point-like K-theory classes concentrated at the origin. This is a wrapped brane on a two-cycle, which is the cycle that one can blow-up to resolve the conifold. Other non-commutative geometries closely related to this system have also been analyzed with these tools in [57].

Another situation in which conifolds make their appearance in non-commutative geometry is when we consider a deformed $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion [18, 35].

The algebra associated to this deformed orbifold is given by

$$\begin{aligned} xy + yx &= \zeta_1 \\ xz + zx &= \zeta_2 \\ zy + yz &= \zeta_3 \end{aligned} \tag{5.5}$$

The variables describing the center are $X = x^2$, $Y = y^2$, $Z = z^2$ and $W = xyz + \alpha x + \beta y + \gamma z$, with α, β, γ determined by the parameters ζ_i. The variables satisfy

$$XYZ = W^2 \,, \tag{5.6}$$

when $\zeta_i = 0$, and one can show that the above deformation leads to a space with one conifold singularity. On analyzing the algebra at the conifold point we find two irreducible one-dimensional representations. This is, we have brane fractionation at the singularity. Locally this is the same conifold as we had before, so one would guess that one should count one local blow-up mode.

However, when seen globally, the two sphere associated to the shrunken cycle is a torsion class in homology, see the discussion in [47]. It follows that the Kähler deformation associated to it is obstructed, because the blow-up cycle would be an algebraic submanifold which is torsion and would be calibrated by the Kähler form. This is, it would be minimizing the volume in it's homology class for any wrapping of the cycle and one arrives at a contradiction. This is peculiar, but it explains why these singularities cannot be blown up in the $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold with discrete torsion, as first shown in [18]. In particular,since the branes do fractionate the blow-up mode should appear, but it is massive and does not lead to new moduli.

Other such spaces constructed for deformed orbifolds with discrete torsion for $\mathbb{Z}_n \times \mathbb{Z}_n$ have the peculiar property that there is a locus where the representations become reducible and one dimensional [45] and there is no singularity in the commutative space to resolve.

6. Regularity, resolutions and intersection theory

String theory compactified on orbifolds can be considered a smooth theory, as the CFT associated to an orbifold is non-singular [13, 14], and moreover one can compute the spectrum and correlation functions of the theory completely in certain situations.

We would like to see that in the non-commutative sense, an orbifold is a smooth space.[3] Obviously, we still have singularities and D-brane fractionation at singularities as viewed from a commutative geometry point of view, as our examples have shown so far. We want to understand how the non-commutative geometry 'resolves' the singularities and what definition of smoothness is appropriate. This should be a local question, and moreover the non-commutative geometry is derived from D-branes, so this question should have a D-brane answer.

Essentially what we envision here is an extension of Douglas' ideas [20] to singular spaces, and non-commutative geometries. The essential result is that one must take care in extending the formalism to non-commutative algebras in that we must distinguish branes as left modules, etc. A nice notion of smoothness will emerge, that of *regularity* of the algebra.

The first thing we need to do then, is give a formulation of (extended) D-branes. For the time being, we will assume that the D-branes are BPS and can be realized holomorphically. A D-brane should be associated with a K-theory class, and this means a formal difference of two vector bundles. On a non-singular commutative space a vector bundle is a locally free module over the algebra of functions (that is, it has constant rank). Thus a holomorphic vector bundle on a complex manifold can be represented by a locally free sheaf globally.

Since not all D-branes are space filling, it is natural to replace locally free sheaf by a coherent sheaf; also when one considers branes embedded into branes this appears naturally [61]. A coherent sheaf S is locally the cokernel of a map between two locally free modules

$$\mathcal{A}^n \to \mathcal{A}^m \to S \to 0\,, \tag{6.1}$$

[3]One should compare this with the approach of E. Sharpe via stacks [58, 59]. Essentially both approaches give the same type of information[60], but we believe our approach is more convenient for computations.

where the above is an exact sequence. If the complex dimension of the manifold is d (and the manifold is non-singular) any coherent sheaf admits (locally) a free resolution of length $d+1$ [31]. That is, we can find holomorphic vector bundles (locally free sheaves) $F^0, \dots, F^d$, such that there is an exact sequence of sheaves

$$0 \to F^d \to F^{d-1} \to \cdots \to F^0 \to S \to 0\,. \tag{6.2}$$

With this construction we can write the K theory class of S as

$$K(S) = K(F^0) - K(F^1) + K(F^2) - K(F^3) + \cdots + (-1)^d K(F^d) \tag{6.3}$$

and in principle we should be able to replace S by it's resolution. When this is done we are starting to see the appearance of homological algebra, which is the study of the topological properties of the complexes of objects in the category of sheaves. To include both branes and antibranes it is more convenient then to introduce the derived category of coherent sheaves [20]. This category is built out of complexes of coherent sheaves with some identifications, see [29, 28] for more details. Recent work on this subject has also appeared in [62, 63, 64], but they focus on either commutative geometry tools or linear sigma models for these problems.

In the non-commutative setting we want to do the same thing. There are a few differences however. We will still define a coherent sheaf as a cokernel of a map between two locally free left modules, the map being a left module map

$$\mathcal{A}^n \to \mathcal{A}^m \to S \to 0\,. \tag{6.4}$$

The K-theory we are after is the standard algebraic K-theory, defined by the projective left modules of the algebra [21]. The above map should be considered as a map of left modules.

When we are studying local geometries which are just $n \times n$ matrices over the local commutative ring the above consoderation is enough. For infinite matrices we have to be more careful and we have to replace the above by

$$P^1 \to P^0 \to S \to 0\,, \tag{6.5}$$

where the P^i are projective of finite type. The two definitions are identical for the $n \times n$ matrices, but the second is better suited for the infinite dimensional case. The ring as a left module is of infinite rank in the case of infinite matrices.

Now, we have coherent sheaves of left modules[4] of the ring as our candidates for extended D-branes. The next thing we need is a definition of smoothness. The proper definition is related to the existence of projective resolutions. A non-commutative

[4]Originally it was suggested to use bimodules in [11] because the representation theory themselves are bimodules, but not the space on which they act on the left. A bimodule is more appropriate to study strings between two D-branes, hence the confusion.

ring is defined to be regular at a point p if every coherent sheaf (localized at p) admits a projective resolution (of finite length $d+1$), that is, we have

$$0 \to P^d \to P^{d-1} \to \cdots \to P^0 \to S \to 0 \tag{6.6}$$

and the projective left-modules are the equivalent of vector bundles over the non-commutative space, with d the complex dimension of the manifold. Thus, we have a condition to check for our examples. This is a good definition because the K-theory of the space is constructed from projective modules. Thus in this setting every brane has a well defined K-theory charge. This condition also seems to be required in order to be able to construct the K-theory of the string background directly from the derived category, (see [65], ch. 5 for more background). There is a subtle technical point to be made here. On analytic manifolds, vector bundles are both projective and injective, because once one imposes a metric on the bundle one can take orthogonal complements, but this is not true in the algebraic category of sheaves. Injective resolutions always exist, and that is how one usually defines the global sections functor [31]. However injective sheaves are notoriously large (they are not finitely generated) and at least in this sense one would be forced into considering an infinite number of brane anti-brane pairs, see however [66, 67]. Hence, these cannot be interpreted in terms of K-theory in a straightforward manner. Locally projective (finite) resolutions do not always exist. Indeed the commutative (singular) space of the center fails to have this property and is therefore non-smooth in this sense as well.

When we are at a non-singular point in the commutative sense, the local non-commutative algebra is Morita equivalent to the local commutative algebra. The local commutative algebra is regular, so any (finitely presented) module over it (coherent sheaf) admits locally a projective resolution. Because Morita equivalence establishes an isomorphism on the categories of modules over the rings, one gets via this ismorphism a local projective resolution for any sheaf over the non-commutative ring. Therefore the non-commutative ring is also regular at these points. Thus the only places where we need to check the regularity condition is exactly at the singularities.

Here we will analyze the case $\mathbb{C}^2/\mathbb{Z}_n$, and we will concentrate only on the compactly supported K-theory classes. That is, we will consider a torsion sheaf at the origin, and all such are fractional branes.

A single fractional brane is an irreducible representation of the algebra, and as a complex vector space has dimension one. As $\sigma^n = 1$, there are n natural candidates for projective modules over the local algebra, and they are related to the n possible projection operators $\pi_k = 1/n \sum \omega^{ki}\sigma^i$ on the group algebra. These are the only n non-trivial projectors at the origin, and they cannot be made homotopic to one another, as they act by different values on the irreducible representation at the origin. Away from the origin, one of x, y is invertible, as their n-th power is, and we can use conjugation by this invertible element to turn the different projectors into each other.

Thus, as our local left modules we take $M_i = \mathcal{A}\pi_i$. We can use these to produce a sequence for each irreducible representation of the algebra S_k, as follows

$$0 \to \mathcal{A}\pi_k \to \mathcal{A}\pi_{k-1} \oplus \mathcal{A}\pi_{k+1} \to \mathcal{A}\pi_k \to S_k \to 0 \tag{6.7}$$

with the Koszul complex on each of the resolutions. That is, the module maps are given by multiplication with x, y on the right. This can be understood in the above sequence because each of the modules is a submodule of the ring $\mathcal{A}$ itself, multiplying on the right by x, y gives us new elements in $\mathcal{A}$ which belong to different projective submodules. Moreover multiplication on the right commutes with multiplication on the left, so these maps are left module morphisms. To be more explicit, if we have elements $a \in \mathcal{A}\pi_k, (b_1, b_2) \in (\mathcal{A}\pi_{k-1}, \mathcal{A}\pi_{k+1}$ of each of the modules the maps are given by

$$\begin{aligned} a &\to (ay, ax) \\ (b_1, b_2) &\to b_1 x - b_2 y \,. \end{aligned} \tag{6.8}$$

Thus a fractional brane admits a projective resolution. A general proof that the ring is regular is beyond the scope of this paper (such a proof will be presented elsewhere [68]).

Now that we have some sense in which we can call the orbifold smooth from the non-commutative geometric point of view, we need to interpret the module maps in the above exact sequences. For the fractional branes, it is clear that the module maps give isomorphisms on the fibers away from the origin. These isomorphisms of the fibers at infinity are familiar from K-theory with compact support, and they have been interpreted as tachyons by Harvey and Moore [69]. In our case notice that the tachyon profile is analytic, and at first sight this would seem like a disaster, as the energy associated to this configuration would be infinite. However, we are analyzing complex vector bundles, and for these, the structure group is complexified. The fact that the functions are divergent at infinity is a gauge artifact, as we are not using a hermitian connection on the bundle. If this is done, the above construction is just a continuous profile which is bounded at infinity. Since we have not mentioned metrics at all, introducing a metric now makes us consider hermitian Yang-Mills bundles instead of holomorphic vector bundles (sheaves), but the two moduli spaces one can compute should be related. In this language one would talk of complexes of differential algebras instead, see for example [20, 70, 62, 71] and one would use the cohomology groups of $\bar{\partial}$ twisted by a holomorphic vector bundle. These results should be equivalent [72].

6.1 Counting open strings and quiver building

Given two D-branes, one wants to consider counting the number of open massless string states between them. This counting should be invariant under Morita equivalence, should be naturally associated to the coherent sheaf (or to an element of the derived category which represents it), and it should come from a homological framework.

The invariance under Morita equivalence is obviously a symmetry we have been considering already, and we have shown that Morita invariant algebras represent the same space, thus the topological data we can extract from D-branes should be the same on each possible representation. Obviously we want an algebraic machine with which to calculate these, and the BRST operator of open strings[20, 62] suggests that the result should be homological.

Naturality here is interpreted in the functorial language. Since we are considering two left modules of the algebra $\mathcal{A}$ and maps that preserve this condition (the projective resolutions), the natural choice is to use $hom_{\mathcal{A}}(A, B)$ for two modules A, B. This is invariant under Morita equivalence, as two rings are Morita equivalent if their categories of modules are isomorphic and this includes morphisms of modules. Since hom is not an exact functor one uses instead the derived functor of hom. Thus the groups which are important for our computation are the $\mathcal{E}xt^i(A, B)$, which can be extended to the derived category. Although for our computations this is all we need, in general it seems to be important to define the category of branes by also counting the ghost number in the open string fields [20, 71, 62, 70], that is, one should keep all of the information in the grading of the $\mathcal{E}xt^i$. The contribution to the intersection index will depend on $i \mod 2$ alone however.

The definition of the $\mathcal{E}xt$ groups can be done using the projective resolutions of the module A, $\cdots \to P_0 \to A \to 0$. One obtains a complex

$$0 \to hom(P_0, B) \to hom(P_1, B) \to \cdots \tag{6.9}$$

The $\mathcal{E}xt^i(A, B)$ groups are the homology groups of the above complex (see [29, 73] for example), and they are independent of the choice of the resolution. Thus one can calculate these explicitly if one knows the projective resolution of an object.

The group $\mathcal{E}xt^1(A, B)$ is special, as it counts locally the space of deformations

$$0 \to A \to C \to B \to 0 \tag{6.10}$$

for gluing two branes A, B to form a third D-brane which is a (marginally) bound state of the two branes, $C \sim A \oplus B$. This is exactly how one understands the process of brane fractionation to make a bulk brane.

Let us take for example $\mathbb{C}^2/\mathbb{Z}_n$. We have already computed the projective resolution of all of the fractional branes in (6.7). Now we want to understand the intersection product.

Thus consider the $\mathcal{E}xt^i(S_i, S_k)$ groups. Since each of the $\mathcal{A}\pi_i$ are generated by the unique section in degree zero π_i, the image of this element generates the module map. Moreover the S_i are concentrated in degree zero, and the map reduces to a computation in the group algebra, namely $hom(\mathcal{A}\pi_l, S_k) \sim \hom_G(\chi_l, S_k)$, where χ_l is the representation of the group G in degree zero in the module. Since each fractional brane corresponds to a unique irreducible, the dimension of the above vector space is $1, 0$ depending on whether $l = k$ or not. In the *hom* complex one obtains, the maps x, y act by zero, thus the homology of the complex is the complex itself.

We find the following results

$$\begin{aligned} \dim \mathcal{E}xt^0(S_i, S_k) &= \delta_{ik} \\ \dim \mathcal{E}xt^1(S_i, S_k) &= \delta_{i,k-1} + \delta_{i,k+1} \\ \dim \mathcal{E}xt^2(S_i, S_k) &= \delta_{ik}\,. \end{aligned} \tag{6.11}$$

Notice that the quiver diagram of the singularity is captured by $\mathcal{E}xt^1(A, B)$. In considering topological string states these are the states that correspond to the vector particles, while $\mathcal{E}xt^0$ counts the possibility of tachyon condensation between branes and anti-branes.

The intersection number between two branes is the Euler characteristic of the above complex (perhaps up to a sign). We obtain that the self intersection of the fractional branes is -2, and that the matrix of Euler characteristics is exactly the Cartan matrix of the extended affine A_n algebra. This is also the intersection form of the resolved orbifold by blow-ups, where each fractional brane wraps one of the exceptional divisors, plus a brane that wraps all of them with the opposite orientation (the extended root of the system). When the cycles are blown-down, these states are mutually BPS.

The advantage of formulating the problem in terms of homological algebra is that the above procedure gives us a recipe for computing these numbers in a general non-commutative geometry singularity which is not of the orbifold type. We can compute these numbers for the fractional branes in the Klebanov-Witten conifold and we obtain the quiver diagram consisting of two nodes with two arrows between them, which contains only the fields which are not lifted by the mass deformation, exactly reproducing the field theory data of the CFT.

There is another point to make which is useful. This intersection formula between fractional branes is symmetric if the complex dimension of the space we are orbifolding is even, and antisymmetric if the complex dimension of the space is odd [68]. This is exactly what one expects from the topological intersection pairing of the BPS D-branes in the mirror manifold, which are even or odd real cycles depending on the dimension of the Calabi-Yau space.

7. Closed string states

So far we have concentrated on open string states alone, with a brief mention of the center of the algebra as the place where closed strings propagate. The description of backgrounds we are constructing depend on having D-branes on them, and can be mainly considered as open string theories.

Open string theories will contain closed strings, so it is natural to ask if there is a way to construct closed string states from a non-commutative algebra. We will restrict ourselves to topology, and hence to topological string states alone.

As argued in [12] the natural way to introduce closed string states is suggested by AdS/CFT duality [74, 75, 3], where the single trace gauge invariant states of a field theory on D-branes are naturally associated to closed string states in the near horizon geometry. Here, we want to exploit this philosophy and write generic closed string states as 'gauge invariant' single trace operators.

The general construction should also accommodate the notion that closed topological string states are harmonic forms on the space, so it is natural to look for them as traces of non-commutative forms, and to check that they are closed. That is, one would want the analog of deRham currents for a non-commutative algebra. A second point to consider is that topological closed strings should be associated with Chern-Simons couplings to D-branes.

We are mainly interested in point-like D-branes, so we will introduce the couplings to these states as traces of algebra elements in the representation associated to a D-brane. We will require that the values of these traces depend only on the homotopy class (K-theory class) of a D-brane, and as such they are topological.

It is best to work with our familiar example $\mathbb{C}^2/\mathbb{Z}_n$. We have n distinct point-like K-theory classes, one for each node in the quiver diagram of the singularity. They are distinguished by the eigenvalue of σ. Thus it is natural to take the traces

$$\mathrm{tr}(\sigma^k) \tag{7.1}$$

as our candidate closed string states, namely the characters of the group elements. Notice that as the quantum symmetry acts on σ, there is an action of the quantum symmetry on the closed string states. These are then naturally twisted sectors.

Notice that only the one corresponding to $\mathrm{tr}(\sigma^0)$ is untwisted. On a bulk brane, the traces for twisted states give zero, as a bulk brane has one of each irreducible representation of the group algebra. Thus we have n traces, one of which is untwisted. The blown-up ALE space also has n closed forms with compact support, one representing each of the $n-1$ blown up self-dual cycles, and one more for the class of the point.

Traces like $\mathrm{tr}(\sigma^k P(x,y))$ for $k \neq 0$ all vanish outside the singularity and moreover they evaluate to zero on the fractional branes. Thus these don't contribute more states. Other traces like $\mathrm{tr}P(x,y)$ depend on the location of a brane in moduli space, unless the polynomial is of degree zero.

A generalized trace that can also measure extended D-brane charge should be (locally) of the form

$$\mathrm{tr}(a_0 da_1 da_2 \dots da_n) \tag{7.2}$$

for elements of the algebra, and these are exactly the elements of the cyclic homology of the algebra [23]. On bulk points, these will reduce to ordinary elements of deRham cohomology on the local commutative space, via Morita equivalence in cyclic homology.

7.1 Betti Numbers for global non-commutative Calabi Yau spaces

In this section we will give a prescription for computing the Betti numbers of the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with and without discrete torsion. Once these two examples are dealt with, we will give a general recipe for computing the Betti numbers of certain singular Calabi-Yau 3-fold with singularities in codimension 2 with a non-commutative resolution of singularities.

To begin with the computation of the Betti numbers of our main example, we have to deal first with the $\mathbb{C}^2/\mathbb{Z}_2$ singularity, which is the local version of the singularities of codimension two.

We already have the algebra of $\mathcal{A} = \mathbb{C}^2/\mathbb{Z}_2$. String theory predicts that in these singularities we generate as many blow-ups and complex structure deformations as there are massless twisted sectors, in this case one. Blowing up or deforming the singularity we find one square integrable harmonic form of type $H^{1,1}$. The fractional brane in the blow-up becomes a brane wrapped around the resolved cycle, and integrating this (normalized) $(1,1)$ form over the compactly supported brane can serve to count how many branes are wrapping the cycle. This is basically the topological coupling of the D-brane data to the RR potential in string theory, the Chern-Simons coupling. The non-torsion part of the lattice of K-theory can be paired with cohomology with complex coefficients, and the two carry the same information [23].

Thus counting the cohomology classes can be done by computing invariants of K-theory classes. The closed string states are then the dual space for these objects (couplings in the action for open string states), and one should basically think of them as integrating over cycles (D-branes with some excited states on them) in the manifold and producing numbers (the action for the configuration). When we write an expression in terms of traces, this will be implicitly understood.

In our local case, we have brane fractionation at the origin, and we should be able to compute invariants of these point-like classes that are additive in the K-theory classes and vanish for the non-fractional brane. Indeed, the fractional branes correspond to the one-dimensional irreducible representations of the algebra of $\mathbb{C}^2/\mathbb{Z}_2$. The ring at the singularity consists only of the group algebra, and the traces reduce then to characters of the group elements in the associated representations. There are only as many linearly independent traces as there are conjugacy classes of elements in the group G, which is identical to the number of irreducible representations of the group and to the number of closed string twisted sectors.

In our case, there is the trivial trace of 1, corresponding to the trivial representation of the group, and this trace counts the total number of bulk zero branes.[5] There is a second trace, $\mathrm{tr}(\sigma)$. For the two one-dimensional representations at the fixed point, the value of this trace is ± 1, so they wrap the singular cycle with the opposite orientation. The state $\mathrm{tr}(\sigma)$ vanishes on the bulk representation. Thus the support of the state (the points in the non-commutative moduli space where $\mathrm{tr}(\sigma) \neq 0$) consists of the two points at the origin. This is also a twisted sector, as when we act with the quantum symmetry the state changes sign.

Since σ can be chosen globally constant in the representation theory, we want to argue that $d\mathrm{tr}(\sigma) = \mathrm{tr}(d\sigma) = 0$, so that $\mathrm{tr}(\sigma)$ is a cohomology class for d.[6]

We also know that we can blow-up the cycle, and that this twisted class becomes a $(1,1)$ form on the blow-up. The fractional brane should be imagined to be wrapping a 2-sphere cycle on the orbifold [49, 50], and in the trace we are integrating over this cycle. Thus our twisted state should be identified with a $(1,1)$ class.

In terms of traces of elements of the algebra, it is not clear why these should correspond to $(1,1)$ classes. However, their holomorphicity suggests that they should be associated with (k,k) classes for some k [76, 77, 78].

Now, we have these classes locally on any patch of our non-commutative space, and we need to glue them together so that we have global holomorphic forms.

In the case of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ without discrete torsion, we get one such trace for every fixed plane. Notice that the group elements survive the quotient (they are not eliminated by a Morita equivalence), and on each codimension two singularity there is one group element which does not vanish, thus for each fixed plane we have one blow-up mode. This is also the result we get from conformal field theory [18]. Remembering the structure of the cover of the singularity (this is depicted in figure 4), the new class corresponds to $b^0(\mathbb{CP}^1 \oplus \mathbb{CP}^1) - 1$ (the 0-th Betti number), where we subtract the trivial class of the brane in the bulk which is not a new class.

For the case of the orbifold with discrete torsion, we have to remember that we changed variables to the $\tilde{y}$, and that we can eliminate the group variables by Morita equivalence.

In this case, at the singularity the traces that do not vanish are

$$\mathrm{tr}(\tilde{y}_i^k) \tag{7.3}$$

with k odd. Now, when we move the branes on their moduli space, these vary and there is no new $H^{1,1}$ class, as there is no global holomorphic function which is odd in the y on the singularity. This is because our double cover of the singularity is

[5] This coupling measures the total D0-brane charge. On extended branes this is an integral over the whole brane, and it essentially measures the top chern class of the fiber bundle, with perhaps some corrections from geometry, see [7].

[6] Here we are loosely extending d to the singular point.

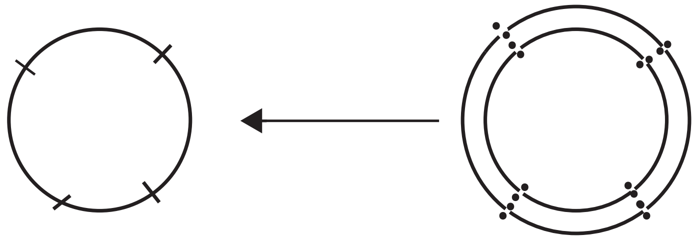

Figure 4: Noncommutative cover of singularity without discrete torsion: the special points correspond to the intersection of two curves of singularities.

Figure 5: Noncommutative cover of singularity with discrete torsion: at the special points the fractional branes have monodromies [47].

connected and compact, thus $b^0(T^2) - 1 = 0$. From string theory this is just as expected, but now we also expect one new $H^{2,1}$ class for each cycle.

However, the double cover of the singular $\mathbb{CP}^1$ is a torus, and we have a natural 1-form on this torus which is it's abelian one-form. This cover is depicted in figure 5. The expression in Weierstrass form for this one-form is

$$\mathrm{tr} dy_i/x_i = \frac{d(\mathrm{tr}(y_i))}{x_i} \tag{7.4}$$

which we can also see has support on the singularity, as elsewhere $\mathrm{tr}(y_i) = 0$. This new one-form is wrapped on the same $(1,1)$ cycle as we had before, so it should correspond to an element of $H^{2,1}$ of the Calabi-Yau manifold. Hence we get a contribution to $b^{2,1}$ from $b^{1,0}$ of the singular set [12].

Notice that now, for the case without discrete torsion, the singular set is fibered non-commutatively by two copies of $\mathbb{CP}^1$, each of which is simply connected, hence there is no contribution to $H^{2,1}$, as there are no holomorphic one-forms on the singular set. This is also in accordance with conformal field theory calculations.

For $b^{2,2}$ of the Calabi-Yau we need to do the same analysis. By Hodge duality, this should be equal to $b^{1,1}$ of the CY manifold. In our case, both the T^2 and the $\mathbb{CP}^1 \oplus \mathbb{CP}^1$ have classes in dimension $(1,1)$ that can contribute.

Now let us discuss more general Calabi-Yaus. A codimension two singular curve contributes to $H^{1,1}$ according to the number of connected components of the non-commutative fibration. One also sees contributions to $H^{2,1}$ by an amount equal to $H^{1,0}(C) - x$, where x is the number of global one-forms shared with the manifold (so if the manifold is simply connected, $x = 0$). We also contribute to $H^{2,2}$ by the same amount as to $H^{1,1}$, mainly because we are imposing Hodge duality by hand, as it holds on the covers of the individual singular cycles. The orbifold cohomology theory indicates that this can be assumed in general [76], and since we are in a Calabi Yau manifold the cohomology classes will not have fractional degree.

For codimension three orbifold singularities in a Calabi-Yau, we get local fractional branes. These singularities contribute to both $H^{1,1}, H^{2,2}$ of the global Calabi-Yau by a number equal to $(H^0 - 1)/2$, and it is also known that they don't contribute to $H^{2,1}$. Usually orbifold points are at walls between distinct phases [79, 80] but all of these have the same Betti numbers and moduli spaces of complex structure deformations, so even though we are at a transition point between topologies the non-commutative results we have obtained are meaningful.

8. Outlook

Strings propagating on singular spaces might not be singular, and here we have made a proposal on how to interpret this fact by exploiting the non-commutativity of open strings. The proposal has two geometries: one non-commutative geometry of the open strings, and a derived commutative geometry of closed strings (in our case defined by the center of the local algebras). The main ingredient in this resolution is the use of the category of D-branes to define the geometry. This simple setup provides a remarkable amount of topological information, namely, one can calculate the intersection theory of branes very generally. One can also explicitly see the existence of closed string twisted sectors at singularities and this is how one usually resolves singularities by closed string fields.

However, the examples we have provided are not general enough to capture all of the possible non-commutative geometries that strings can describe and should be viewed as a first step towards a more general theory. For example, the non-commutative plane is left out of our construction, namely because the algebra is not a finite matrix algebra over the center; this non-commutative geometry would be a fibration over a point.

From a Mirror Symmetry perspective we have only dealt with the B model physics, so one would also be interested in exploring how to realize these ideas in the A model as well, and in particular how to calculate stringy corrections to different quantities in this setup, see for example [81]. It would also be interesting to extend these non-commutative ideas to the Landau Ginzburg phases of string theory.

However, given our particular realization of geometry it is also very difficult to introduce a worldsheet string action. Should one use a non-commutative sigma model in the spirit of matrix strings [82]? Or should we use instead a commutative sigma model action with a prescription for dealing with the singularities? It has been shown that writing matrix theories on general backgrounds is difficult [83]; however,

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by restricting ourselves to topological string theories we might be better off, as the number of string fields that one needs to consider becomes finite and the theory does not depend on the exact properties of the metric.

The description we have given should only be applicable to situations where the singularity can be resolved perturbatively in the string theory. This is particularly evident because closed string states and D-branes wrapping cycles are treated in a completely different fashion. It is unclear if non-commutative geometry will play any role at all at the conifold transition.

These and other issues are currently under investigation.

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